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Separation Property for the Rigid-Body Attitude Tracking Control Problem

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Quaternion-based proportional-derivative controllers for rigid-body attitude dynamics provide globally stabilizing solutions to both set-point regulation and trajectory tracking problems. Because the quaternion vector, or for that matter, any other attitude representation, can never be directly exactly measured, proportional-derivative controllers are invariably implemented under the assumption that the attitude error is available from suitable observers whose estimates converge sufficiently fast to the corresponding true attitude. To compound the situation, given the nonlinearities within the governing dynamics, most existing attitude observers can at best be proven to provide only asymptotic (i.e., nonexponential) convergence for the attitude estimation errors. This has the serious consequence that closed-loop stability assurances provided by classical proportional-derivative control laws no longer remain valid when the true attitude errors are replaced by their corresponding estimates. In this paper, we present a new quaternion-based attitude tracking controller that guarantees global asymptotic stability for the closed-loop dynamics while adopting an observer to generate the quaternion-based attitude estimates. We show that the state feedback control law and the estimator can be independently designed so that closed-loop stability is maintained even when they are combined. Accordingly, a separation property is established for the rigid-body attitude tracking problem, the first such result to our best knowledge. The crucial step in our stability analysis involves introduction of a novel class of strict Lyapunov functions whose time derivatives contain additional negative terms that help dominate the error terms arising due to the attitude observer implementation. Detailed proofs and numerical simulation examples are presented to help illustrate all the technical aspects of this work.

Nomenclature

$\boldsymbol{B}$, $\hat{\boldsymbol{b}}$	=	body-fixed frame and basis
$C(\cdot)$	=	direction cosine matrix
$\boldsymbol{E}$, $\hat{\boldsymbol{e}}$	=	estimated body frame and basis
$E(\cdot)$	=	quaternion operator
$I_{3\times 3}$	=	3×3 identity matrix
J	=	inertia matrix
k_p, k_v	=	control gain
$\boldsymbol{N}$, $\hat{\boldsymbol{n}}$	=	inertial frame and basis
$\boldsymbol{q}$, $\hat{\boldsymbol{q}}$	=	quaternion and estimated quaternion
$\boldsymbol{R}$, $\hat{\boldsymbol{r}}$	=	reference frame and basis
$S(\cdot)$	=	matrix cross-product operator
$\boldsymbol{u}$	=	control torque input
V, V_c, V_o	=	Lyapunov-like candidate functions
$\boldsymbol{x}$, $\dot{\boldsymbol{x}}$	=	known input signal for observer and its time derivative
$\boldsymbol{y}$, $\hat{\boldsymbol{y}}$	=	attitude measurement and estimated measurement
$\boldsymbol{z}$	=	multiplicative quaternion attitude error between body-fixed and estimated body frame
α	=	filter gain
γ	=	observer gain
$\delta\boldsymbol{q}$	=	quaternion error between body-fixed and reference frame
$\delta\hat{\boldsymbol{q}}$	=	quaternion error between estimated body and reference frame
$\delta\boldsymbol{\omega}$	=	angular velocity error
$\delta\hat{\boldsymbol{\omega}}$	=	angular velocity error with regard to estimated body frame
$\boldsymbol{\omega}$	=	angular velocity

Subscripts

f	=	filter state
o	=	scalar part of the quaternion
r	=	reference trajectory for the tracking system
$\boldsymbol{v}$	=	vector part of the quaternion

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I. Introduction

PROPORTIONAL-derivative (PD) type attitude control systems for rigid spacecraft, i.e., rigid-body rotational dynamics modeled by Euler's equations together with a suitable attitude parameterization, have been extensively studied during the past decades [1–9]. More specifically, if the spacecraft is endowed with three independent torque actuators, then a complete (global) solution is available for both set-point and trajectory tracking control problems assuming availability of the full-state vector composed of the body angular rates and the globally valid (nonsingular) quaternion vector. It is a well-documented fact that the set of the special group of rotation matrices that describe body orientation in three dimensions $SO(3)$ is not a contractible space and hence quaternion-based formulations do not permit globally continuous stabilizing controllers [1,10]. In this sense, we adopt the standard terminology/notion of (almost) global stability for this problem to imply stability over an open and dense set in $SO(3)$ as is usually seen in literature dealing with this problem [5].

When it comes to implementation of the PD-type controllers, virtually all existing attitude control designs assume direct availability of the attitude (quaternion) vector for feedback implementations. However, the practical reality is that there exists no physical sensor that permits direct and exact measurement of the quaternion vector or any other representation of the body attitude. Motivated by this important practical consideration, there indeed exists a separate thread of research that bears several rich results addressing the development of attitude estimators (observers) formulated in terms of various attitude parameterizations [11]. These estimators are driven by measurements from rate gyros, sun sensors, star sensors, earth sensors, magnetometers, and a host of other sensor candidates; essentially depending upon the particular application [12–17] and their convergence proofs provide, at best, asymptotic (nonexponential) convergence to the true (actual) body attitude.

Given that the governing dynamics are nonlinear and time-varying, any closed-loop stability result obtained from the PD-type controller assuming exact measurement of the quaternion vector needs to be reestablished all over again once the estimated quaternion is adopted. Therefore, a question of great theoretical and practical importance arises as to what can be said about closed-loop stability when the PD control implementation is modified so that the feedback uses attitude estimates generated by observers instead of the actual (true) attitude variables. Because the separation property [18] does not hold in general for nonlinear systems, this remains an open problem to our best knowledge for which no satisfactory solution currently exists. A key technical difficulty on this front is the fact that the closed-loop stability for the control problem is established through energy-type Lyapunov-like functions that are not strict [1], i.e., their derivatives are only negative semidefinite involving only angular rate related terms. In such a setting, technical arguments involving LaSalle invariance (for stabilization or set-point regulation) and/or Barbalat's lemma (for trajectory tracking) are the common recourse for completing the closed-loop stability analysis. Not surprisingly, due to the nonstrict nature of the Lyapunov-like functions in the full-state feedback control analysis, one encounters the formidable uniform detectability obstacle [19] whenever PD-based control designs are sought to be combined with attitude (quaternion) estimators.

In this paper, we propose a novel methodology for design of a strict Lyapunov-like function in terms of the quaternion parameterization of the attitude such that the PD-type feedback controller ensures almost global closed-loop stability. Further, by making use of the proposed strict Lyapunov-like function construction, and thereby avoiding the detectability obstacle, we present a rigorous stability analysis for the PD-based controller when it is combined with a suitable attitude quaternion observer (estimator).

The fundamental contribution of this paper is that if the reference signal for the attitude estimator possesses an easily verifiable persistence of excitation condition and the initial condition of the quaternion estimator $\hat{\boldsymbol{q}}(0)$ does not lie inside a hyperplane normal to the initial condition of the actual/true attitude quaternion $\boldsymbol{q}(0)$, i.e., $\hat{\boldsymbol{q}}^T(0)\boldsymbol{q}(0) \neq 0$, then we prove global stability and asymptotic convergence for the attitude tracking error with respect to any commanded attitude reference trajectory. Our results essentially imply that separate designs of the full-state feedback controller and the attitude estimator may be readily combined to ensure overall closed-loop stability. To our best knowledge, this is the first-ever demonstration of separation property in the setting of the rigid-body attitude tracking control problem.

The paper is organized as follows. In Sec. II, we develop a new full-state feedback controller derived through a strict Lyapunov function candidate and then prove the asymptotic stability of the closed-loop error dynamics. In Sec. III, we develop a new attitude observer because the true attitude values are assumed unavailable to the controller. The estimation error dynamics are derived based on the quaternion observer states. Section IV presents the main results of this paper for the combined estimator-controller closed-loop attitude tracking system. In Sec. V, numerical simulations are presented to describe the performance of the proposed theoretical results. We complete the paper in Sec. VI with appropriate concluding statements.

II. Full-State Feedback Control with True Attitude Values

First, we develop a full-state feedback controller for the attitude tracking control system based on the true attitude values. The proposed control method provides a strict Lyapunov candidate function in terms of the filtered angular velocity and the attitude values explicitly for all initial values (singularity [9] depending on the initial condition is removed). The introduced angular velocity filter is a simple low-pass filter and used only for the stability analysis, i.e., the implementation of the proposed control scheme does not depend on the angular velocity filter states. It will be clear in the later part of this section. To represent the attitude of the system, quaternion is adopted for nonsingular representation through all possible rotation angles [11]. We denote the true attitude of the system as $\boldsymbol{q}$. Each quaternion vector obviously has four parameters where the first element, the scalar part of the quaternion, is represented by the subscript "o" and the rest of them make up the vector part of the quaternion which is represented by the subscript "$\boldsymbol{v}$" with a bold face character, i.e., $\boldsymbol{q} = [q_o, \boldsymbol{q}_v]^T$. We also specify $\boldsymbol{q}_r$ to denote the reference attitude trajectory in terms of the quaternion parameterization.

The direction cosine matrix $C(\cdot)$ is considered as a mapping from the unit quaternion space $\mathbb{R}^4$ to the proper orthogonal matrix space $SO(3)$ (i.e., $C^TC = CC^T = I_{3\times3}$, and $\det[C(\cdot)] = 1$). In the following development, three frames are used to denote the relative attitude representation associated with $C(\cdot)$ as follows

$$\boldsymbol{N} \overset{q}{\to} \boldsymbol{B} \Rightarrow \hat{\boldsymbol{b}} = C(\boldsymbol{q})\hat{\boldsymbol{n}} \qquad \boldsymbol{N} \overset{q_r}{\to} \boldsymbol{R} \Rightarrow \hat{\boldsymbol{r}} = C(\boldsymbol{q}_r)\hat{\boldsymbol{n}}$$
$$\boldsymbol{R} \overset{\delta q}{\to} \boldsymbol{B} \Rightarrow \hat{\boldsymbol{b}} = C(\delta\boldsymbol{q})\hat{\boldsymbol{r}}$$

where $\hat{\boldsymbol{b}}$, $\hat{\boldsymbol{r}}$, and $\hat{\boldsymbol{n}}$ represent the unit-vector triads in the body frame $\boldsymbol{B}$, the reference frame $\boldsymbol{R}$, and the inertial frame $\boldsymbol{N}$, respectively. Because $C(\boldsymbol{q})$ rotates $\hat{\boldsymbol{n}}$ (inertial frame $\boldsymbol{N}$) to $\hat{\boldsymbol{b}}$ (body-fixed frame $\boldsymbol{B}$) and $C(\boldsymbol{q}_r)$ rotates $\hat{\boldsymbol{n}}$ (inertial frame $\boldsymbol{N}$) to $\hat{\boldsymbol{r}}$ (reference frame $\boldsymbol{R}$), the combined rotation from $\hat{\boldsymbol{r}}$ to $\hat{\boldsymbol{b}}$ can be represented by the following

$$C(\delta\boldsymbol{q}) = C(\boldsymbol{q})C^T(\boldsymbol{q}_r) \tag{1}$$

which leads to the definition of multiplicative quaternion attitude error $\delta\boldsymbol{q}$.

In the sequel, for the sake of notational simplicity, the time argument t is left out except at places where it is noted for emphasis.

The kinematic evolution equation for quaternion $\boldsymbol{q}$ is expressed as

$$\dot{\boldsymbol{q}} = \frac{1}{2}E(\boldsymbol{q})\boldsymbol{\omega} \tag{2}$$

where $\boldsymbol{\omega}$ is the angular velocity with respect to the body-fixed frame $\boldsymbol{B}$ and the quaternion operator $E(\cdot)$:$\mathcal{R}^4 \to \mathcal{R}^{4\times3}$ is defined by

$$E(\boldsymbol{q}) = \begin{bmatrix} -\boldsymbol{q}_v^T \\ q_o I_{3\times3} + S(\boldsymbol{q}_v) \end{bmatrix} \tag{3}$$

Here, the matrix operator $S(\cdot)$ denotes the skew-symmetric matrix that is equivalent to the vector cross product operation as follows [20]

$$S(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \quad \forall\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \tag{4}$$

The time evolution of the body angular velocity is described by the well-known Euler differential equation for rotational motion and is given by

$$J\dot{\boldsymbol{\omega}} = -S(\boldsymbol{\omega})J\boldsymbol{\omega} + \boldsymbol{u} \tag{5}$$

where J is the 3×3 symmetric positive definite inertia matrix and $\boldsymbol{u}$ is the control torque vector.

In terms of the direction cosine matrix $C(\cdot)$, the analogous matrix version of Eq. (2) can be derived as follows [20]

$$\frac{d}{dt}C(\boldsymbol{q}) = -S(\boldsymbol{\omega})C(\boldsymbol{q}) \tag{6}$$

which of course is the well-known Poisson differential equation. The true angular velocity error $\delta\boldsymbol{\omega}$ is defined through

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega} - C(\delta\boldsymbol{q})\boldsymbol{\omega}_r \tag{7}$$

where $\boldsymbol{\omega}_r(t)$ represents the bounded reference angular velocity with bounded time derivatives and is prescribed with respect to the reference frame $\boldsymbol{R}$.

To derive the governing dynamics for the attitude error $C(\delta\boldsymbol{q})$, the following steps are taken starting from Eq. (1)

$$\begin{aligned}\frac{d}{dt}C(\delta\boldsymbol{q}) &= -S(\boldsymbol{\omega})C(\boldsymbol{q})C^T(\boldsymbol{q}_r) + C(\boldsymbol{q})C^T(\boldsymbol{q}_r)S(\boldsymbol{\omega}_r)\\ &= -S(\boldsymbol{\omega})C(\delta\boldsymbol{q}) + C(\delta\boldsymbol{q})S(\boldsymbol{\omega}_r)\\ &= -S(\boldsymbol{\omega})C(\delta\boldsymbol{q}) + S[C(\delta\boldsymbol{q})\boldsymbol{\omega}_r]C(\delta\boldsymbol{q})\\ &= -S[\boldsymbol{\omega} - C(\delta\boldsymbol{q})\boldsymbol{\omega}_r]C(\delta\boldsymbol{q})\\ &= -S(\delta\boldsymbol{\omega})C(\delta\boldsymbol{q})\end{aligned} \tag{8}$$

wherein the identity $C(\delta\boldsymbol{q})S(\boldsymbol{\omega}_r) = S[C(\delta\boldsymbol{q})\boldsymbol{\omega}_r]C(\delta\boldsymbol{q})$ has been employed which can be easily proved by recognizing the invariance of the vector cross product with respect to rigid rotation in three dimensions as follows

$$\begin{aligned}C(\delta\boldsymbol{q})S(\boldsymbol{\omega}_r)\boldsymbol{v} &= C(\delta\boldsymbol{q})(\boldsymbol{\omega}_r \times \boldsymbol{v})\\ &= C(\delta\boldsymbol{q})\boldsymbol{\omega}_r \times C(\delta\boldsymbol{q})\boldsymbol{v} = S[C(\delta\boldsymbol{q})\boldsymbol{\omega}_r]C(\delta\boldsymbol{q})\boldsymbol{v}, \quad \forall\ \boldsymbol{v} \in \mathbb{R}^3\end{aligned} \tag{9}$$

The attitude matrix version of the tracking error kinematics are represented by Eq. (8). Differentiating Eq. (7) along Eqs. (5) and (8) provides the angular rate tracking error dynamics for the rigid-body rotational motion. Consequently, the overall attitude tracking error dynamics and kinematics (expressed in terms of the quaternion attitude tracking error $\delta\boldsymbol{q}$) are as follows [20]

$$\delta\dot{\boldsymbol{q}} = \frac{1}{2}E(\delta\boldsymbol{q})\delta\boldsymbol{\omega} \tag{10}$$

$$J\delta\dot{\boldsymbol{\omega}} = -S(\boldsymbol{\omega})J\boldsymbol{\omega} + \boldsymbol{u} + J\bar{\boldsymbol{\eta}} \tag{11}$$

wherein the quantity $\bar{\boldsymbol{\eta}}$ is defined for notational convenience to represent the following

$$\bar{\boldsymbol{\eta}} = S(\delta\boldsymbol{\omega})C(\delta\boldsymbol{q})\boldsymbol{\omega}_r - C(\delta\boldsymbol{q})\dot{\boldsymbol{\omega}}_r \tag{12}$$

We note here that regulation of $\lim_{t\to\infty}[\delta\boldsymbol{q}_v(t), \delta\boldsymbol{\omega}(t)] = 0$ obviously also goes on to achieve our desired attitude tracking control objective.

We now present the following proposition.

Proposition 1 (full-state feedback). For the attitude tracking system given by Eqs. (10–12), consider the control torque $\boldsymbol{u}$ to be computed as follows

$$\begin{aligned}\boldsymbol{u} &= -k_v J\delta\boldsymbol{\omega} + S(\boldsymbol{\omega})J\boldsymbol{\omega} - J\bar{\boldsymbol{\eta}}\\ &\quad - k_p J\left(\alpha\delta\boldsymbol{q}_v + \frac{1}{2}[\delta q_o I_{3\times3} + S(\delta\boldsymbol{q}_v)]\delta\boldsymbol{\omega}\right)\end{aligned} \tag{13}$$

where $k_p > 0$, $k_v > 0$, $\alpha = k_p + k_v$. Then all closed-loop signals remain bounded and the attitude tracking control objective $\lim_{t\to\infty}[\delta\boldsymbol{q}_v(t), \delta\boldsymbol{\omega}(t)] = 0$ is achieved globally (for all possible initial conditions) and asymptotically.

Proof. For the sake of analysis, we introduce an angular velocity filter state $\boldsymbol{\omega}_f$ governed through the stable linear differential equation

$$\dot{\boldsymbol{\omega}}_f = -\alpha\boldsymbol{\omega}_f + \delta\boldsymbol{\omega}, \quad \text{any } \boldsymbol{\omega}_f(0) \in \mathcal{R}^3 \tag{14}$$

Differentiating both sides of the filter dynamics in Eq. (14) followed by substitution of the angular velocity error dynamics, Eq. (11), along with Eq. (13), we have

$$\frac{d}{dt}[\dot{\boldsymbol{\omega}}_f + k_v\boldsymbol{\omega}_f + k_p\delta\boldsymbol{q}_v] = -\alpha[\dot{\boldsymbol{\omega}}_f + k_v\boldsymbol{\omega}_f + k_p\delta\boldsymbol{q}_v] \tag{15}$$

which obviously has the following analytical solution

$$\dot{\boldsymbol{\omega}}_f = -k_v\boldsymbol{\omega}_f - k_p\delta\boldsymbol{q}_v + \boldsymbol{\varepsilon}e^{-\alpha t} \tag{16}$$

where the exponentially decaying term $\mathbf{w}(t) \doteq \boldsymbol{\varepsilon}e^{-\alpha t}$ on the right-hand side of the preceding equation is characterized by

$$\dot{\mathbf{w}} = -\alpha\mathbf{w}; \qquad \mathbf{w}(0) = \boldsymbol{\varepsilon} = \{\dot{\boldsymbol{\omega}}_f(0) + k_v\boldsymbol{\omega}_f(0) + k_p\delta\boldsymbol{q}_v(0)\} \tag{17}$$

Next consider the following lower-bounded Lyapunov-like candidate function V_t given by

$$V_t = \frac{1}{2}\boldsymbol{\omega}_f^T\boldsymbol{\omega}_f + [(\delta q_o - 1)^2 + \delta\boldsymbol{q}_v^T\delta\boldsymbol{q}_v] + \frac{\lambda}{2}\mathbf{w}^T\mathbf{w}; \quad \lambda > \max\left(\frac{1}{k_p^2}, \frac{1}{k_v^2}\right) \tag{18}$$

By taking the time derivative of V_t along trajectories generated by Eqs. (10), (16), and (17), we have the following

$$\begin{aligned}\dot{V}_t &= \boldsymbol{\omega}_f^T(-k_v\boldsymbol{\omega}_f - k_p\delta\boldsymbol{q}_v + \mathbf{w}) + \delta\boldsymbol{q}_v^T\delta\boldsymbol{\omega} - \lambda\alpha\mathbf{w}^T\mathbf{w}\\ &= -k_v\|\boldsymbol{\omega}_f\|^2 - k_p\boldsymbol{\omega}_f^T\delta\boldsymbol{q}_v + \boldsymbol{\omega}_f^T\mathbf{w} + \delta\boldsymbol{q}_v^T(\dot{\boldsymbol{\omega}}_f + \alpha\boldsymbol{\omega}_f) - \lambda\alpha\mathbf{w}^T\mathbf{w}\\ &= -k_v\|\boldsymbol{\omega}_f\|^2 - k_p\boldsymbol{\omega}_f^T\delta\boldsymbol{q}_v + \boldsymbol{\omega}_f^T\mathbf{w}\\ &\quad + \delta\boldsymbol{q}_v^T(-k_v\boldsymbol{\omega}_f - k_p\delta\boldsymbol{q}_v + \mathbf{w} + \alpha\boldsymbol{\omega}_f) - \lambda\alpha\mathbf{w}^T\mathbf{w}\\ &= -k_v\|\boldsymbol{\omega}_f\|^2 - k_p\|\delta\boldsymbol{q}_v\|^2 - \lambda\alpha\|\mathbf{w}\|^2 + \boldsymbol{\omega}_f^T\mathbf{w} + \delta\boldsymbol{q}_v^T\mathbf{w}\\ &\le -\frac{k_v}{2}\|\boldsymbol{\omega}_f\|^2 - \frac{k_p}{2}\|\delta\boldsymbol{q}_v\|^2 - \frac{\lambda\alpha}{2}\|\mathbf{w}\|^2 - \frac{k_v}{2}\|\boldsymbol{\omega}_f - \frac{1}{k_v}\mathbf{w}\|^2\\ &\quad - \frac{k_p}{2}\|\delta\boldsymbol{q}_v - \frac{1}{k_p}\mathbf{w}\|^2 \le -\frac{k_v}{2}\|\boldsymbol{\omega}_f\|^2 - \frac{k_p}{2}\|\delta\boldsymbol{q}_v\|^2 - \frac{\lambda\alpha}{2}\|\mathbf{w}\|^2 \le 0\end{aligned} \tag{19}$$

which is negative semidefinite and thus the Lyapunov candidate function V_t is *strict* [19] due to presence of nonpositive terms of $\boldsymbol{\omega}_f$ and $\delta\boldsymbol{q}_v$ in $\dot{V}_t$. Moreover, because V_t is lower bounded and monotonic, the integral

$$\int_0^\infty \dot{V}_t(s)\,\mathrm{d}s$$

exists and is finite, which implies $\boldsymbol{\omega}_f$, $\delta\boldsymbol{q}_v \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. This also implies boundedness of $\dot{\boldsymbol{\omega}}_f$, $\ddot{\boldsymbol{\omega}}_f$, and $\delta\dot{\boldsymbol{q}}_v$ from Eqs. (10) and (16). Therefore, using Barbalat's lemma, we can guarantee $\lim_{t\to\infty}[\boldsymbol{\omega}_f(t), \delta\boldsymbol{q}_v(t)] = 0$. Finally, from the stable linear filter Eq. (14), it follows that $\lim_{t\to\infty}\delta\boldsymbol{\omega}(t) = 0$ thereby completing the proof. □

It needs to be emphasized that the filter dynamics from Eq. (14) are required only for analysis as part of the stability proof. This observation is highlighted by the fact that the control law given in Eq. (13) is independent of the filter variables $\boldsymbol{\omega}_f$ which therefore need not be computed in the actual controller implementation. Additionally, it is also always possible to introduce the filter states such that $\mathbf{w}(t) = 0$ for all $t \ge 0$ by letting $\mathbf{w}(0) = 0$ based on the proper choice of initial filter states such that $\boldsymbol{\omega}_f(0) = [\delta\boldsymbol{\omega}(0) + k_p\delta\boldsymbol{q}_v(0)]/k_p$ as can be seen in Eq. (17). Even otherwise, if the filter initial conditions are such that $\mathbf{w}(0) \neq 0$, it is obvious from our demonstration thus far that the exponentially decaying signal $\mathbf{w}(t)$ present in Eq. (16) can be neglected without any loss because it plays no role whatsoever on the stability analysis [21,22]. Accordingly, the exponentially decaying term $\mathbf{w}(t)$ is eliminated hereon from filtered angular velocity error dynamics in Eq. (16) to provide

$$\dot{\boldsymbol{\omega}}_f = -k_v\boldsymbol{\omega}_f - k_p\delta\boldsymbol{q}_v \tag{20}$$

and analogously, the term $\mathbf{w}^T\mathbf{w}/2$ from the Lyapunov-like candidate function in Eq. (18) may be dropped to simplify the stability analysis with no impact whatsoever on the overall asymptotic convergence results assured by the proposed controller.

In contrast to most conventional PD-type quaternion-based formulations for rigid-body attitude control wherein Lyapunov candidate functions are at best negative semidefinite involving only the angular rate error terms, the proposed control method succeeds in introducing an additional negative definite term $-k_p\|\delta\boldsymbol{q}_v\|^2$ in

Eq. (19) which makes V_t a strict Lyapunov function [19]. The importance of the extra negative definite term consisting of the quaternion tracking error is discussed in the sequel.

III. Attitude Observer Design

In many practical applications, full-state feedback attitude control is not readily applicable due to the fact that the true attitude quaternion measurement is typically not available to the controller for feedback purposes. Thus, feedback control signals are sought to be based on the estimated attitude quaternion values that are obtained from the observer (attitude estimator) which in turn is driven by the measured (output) signals. In this section, first we provide a brief presentation of a very recent stability and convergence proof for an attitude estimator [17] and subsequently introduce certain new Lyapunov-like function constructions in the context of the attitude observer convergence analysis that ultimately enable establishment of the nonlinear separation property.

We assume the measurement model and its estimator to be specified in the following fashion

$$\boldsymbol{y}(t) = C[\boldsymbol{q}(t)]\boldsymbol{x}(t) \tag{21}$$

$$\hat{\boldsymbol{y}}(t) = C[\hat{\boldsymbol{q}}(t)]\boldsymbol{x}(t) \tag{22}$$

where the output signal $\boldsymbol{y}$ indicating the true attitude measurements and the estimated measurements $\hat{\boldsymbol{y}}$ in the body-fixed frame. The reference signal $\boldsymbol{x}$ is the unit vector governing the inertial direction of the observation and $\dot{\boldsymbol{x}}$ is assumed to be bounded with time. From a practical standpoint, in three dimensions, the measurement model Eq. (21) is typical for single input-output-type unit-vector measurement sensors such as star trackers, sun sensors, and magnetometers [23]. More specific examples of sensor modeling in attitude determination problems can be found in the literature [24–27]. Discrete-time analogues of the attitude estimation problem routinely arise within the field of spacecraft attitude determination. In such cases, Eq. (21) can be interpreted such that $\boldsymbol{x}$ is the star catalog value with respect to the inertial frame and $\boldsymbol{y}$ is the corresponding star tracker measurement. Vector quantities $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{q}}$ are the interpreted to be estimates of $\boldsymbol{y}$ and $\boldsymbol{q}$, respectively. Obviously, both $\boldsymbol{y}$ and $\hat{\boldsymbol{y}}$ are unit vectors in this model.

In the following proposition, we briefly review the convergence result of an adaptive attitude estimator recently introduced in Akella et al. [17].

Proposition 2 (attitude observer). If the attitude quaternion estimate $\hat{\boldsymbol{q}}(t)$ is updated through

$$\dot{\hat{\boldsymbol{q}}} = \frac{1}{2}E(\hat{\boldsymbol{q}})(\boldsymbol{\omega} + \gamma \boldsymbol{y} \times \hat{\boldsymbol{y}}); \qquad \gamma > 0 \tag{23}$$

subject to the condition that $\hat{\boldsymbol{q}}(0) \in \Psi_s$ where

$$\Psi_s = \{\boldsymbol{\eta} \in \mathbb{R}^4 : \|\boldsymbol{\eta}\| = 1; \boldsymbol{\eta}^T \boldsymbol{q}(0) \neq 0\} \tag{24}$$

then for all unit-vector reference signals $\boldsymbol{x}(t) \in \mathbb{R}^3$ that are not constant with time with bounded derivatives, the following asymptotic convergence condition holds

$$\lim_{t\to\infty} C[\hat{\boldsymbol{q}}(t)] - C[\boldsymbol{q}(t)] = 0 \tag{25}$$

which essentially accomplishes the attitude estimation objective.

Proof. We define a new quaternion error $\boldsymbol{z}(t) = [z_o(t), \boldsymbol{z}_v(t)]^T$ to parameterize $C[\boldsymbol{z}(t)] \doteq C^T[\boldsymbol{q}(t)]C[\hat{\boldsymbol{q}}(t)]$ which has the following identity [20]

$$C^T[\boldsymbol{q}(t)]C[\hat{\boldsymbol{q}}(t)] = I_{3\times 3} - 2z_o(t)S[\boldsymbol{z}_v(t)] + 2S^2[\boldsymbol{z}_v(t)] \tag{26}$$

It is obvious from Eq. (26) that $\boldsymbol{z}_v \to 0$ ($z_o \to \pm 1$) implies $C(\hat{\boldsymbol{q}}) \to C(\boldsymbol{q})$. Now we derive dynamics of $\boldsymbol{z}_v$ by following similar steps described in Eq. (8). Differentiating $C^T(\boldsymbol{q})C(\hat{\boldsymbol{q}})$ with respect to time along with Eqs. (6) and (23) leads to [17]

$$\frac{d}{dt}C(z) = -S[\gamma C^T(\boldsymbol{q})(\boldsymbol{y} \times \hat{\boldsymbol{y}})]C(z) \tag{27}$$

Thus we have the corresponding quaternion dynamics for $\boldsymbol{z}$ given by

$$\dot{\boldsymbol{z}} = \frac{\gamma}{2}E(z)C^T(\boldsymbol{q})(\boldsymbol{y} \times \hat{\boldsymbol{y}}) \tag{28}$$

Further, from Eqs. (21), (22), and (26), the following algebraic identity may be established

$$\boldsymbol{y} \times \hat{\boldsymbol{y}} = C(\boldsymbol{q})[2z_o(\boldsymbol{z}_v \times \boldsymbol{x}) \times \boldsymbol{x} - 2(\boldsymbol{x}^T \boldsymbol{z}_v)(\boldsymbol{z}_v \times \boldsymbol{x})] \tag{29}$$

and thus we can express Eq. (28) as follows

$$\begin{aligned} \dot{z}_o &= \gamma z_o \|\boldsymbol{z}_v \times \boldsymbol{x}\|^2 \\ \dot{\boldsymbol{z}}_v &= \gamma[z_o^2(\boldsymbol{z}_v \times \boldsymbol{x}) \times \boldsymbol{x} - (\boldsymbol{x}^T z)S(\boldsymbol{z}_v)(\boldsymbol{z}_v \times \boldsymbol{x})] \end{aligned} \tag{30}$$

Next, consider the following bounded (from above and below) Lyapunov-like candidate function

$$V_e = \frac{1}{2}\boldsymbol{z}_v^T \qquad \boldsymbol{z}_v = \frac{1}{2}(1 - z_o^2) \tag{31}$$

By taking the time derivative of V_e along the trajectory of $\boldsymbol{z}_v$ in Eq. (30), we obtain the following result

$$\begin{aligned} \dot{V}_e(t) &= \boldsymbol{z}_v^T \dot{\boldsymbol{z}}_v = -\gamma \boldsymbol{z}_v^T \Big[z_o^2(\boldsymbol{z}_v \times \boldsymbol{x}) \times \boldsymbol{x} - (\boldsymbol{x}^T z)S(\boldsymbol{z}_v)(\boldsymbol{z}_v \times \boldsymbol{x})\Big] \\ &= -\gamma z_o^2 \boldsymbol{z}_v^T[(\boldsymbol{z}_v \times \boldsymbol{x}) \times \boldsymbol{x}] = -\gamma z_o^2 \|z_v \times \boldsymbol{x}\|^2 \leq 0 \end{aligned} \tag{32}$$

From $V_e \geq 0$ and $\dot{V}_e \leq 0$, $\lim_{t\to\infty} V_e(t) = V_{e_\infty}$ exists and is finite which leads to the fact that

$$\int_0^\infty \dot{V}_e(t)\,\mathrm{d}t$$

also exists and is finite. Further, from boundedness of $\ddot{V}_e(t)$ [shown by differentiating Eq. (32)], $\dot{V}_e(t)$ is uniformly continuous. As a consequence, we have $\lim_{t\to\infty} \dot{V}_e(t) = 0$. From Eqs. (30) and (32), we notice that $z_o = 0$ defines one of equilibrium manifolds [i.e., $z_o(0) = 0 \Rightarrow z_o(t) = 0$ for all $t \geq 0$]. Because $z_o(t) = \hat{\boldsymbol{q}}^T(t)\boldsymbol{q}(t)$ for all $t \geq 0$ from Eq. (26) [20], $z_o(0) = 0$ is implied from $\hat{\boldsymbol{q}}(0) \notin \Psi_s$. In other words, if $\hat{\boldsymbol{q}}(0) \in \Psi_s$, then $z_o(0) \neq 0$ and from Eq. (30), it is immediately obvious that $z_o(t) \neq 0$ for all $t \geq 0$. In this case, $\lim_{t\to\infty} \dot{V}_e(t) = 0$ implies $\lim_{t\to\infty}[z_v(t) \times \boldsymbol{x}(t)] = 0$. Again from Eq. (30), it means $\lim_{t\to\infty} \dot{\boldsymbol{z}}_v(t) = 0$. Moreover, from uniform continuity of $[z_v(t) \times \boldsymbol{x}(t)]$, we have

$$\lim_{t\to\infty}[z_v(t) \times \boldsymbol{x}(t)] = 0 \Rightarrow \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}[z_v(t) \times \boldsymbol{x}(t)] = 0 \tag{33}$$

which implies that $\lim_{t\to\infty}[\dot{z}_v(t) \times \boldsymbol{x}(t) + z_v(t) \times \dot{\boldsymbol{x}}(t)] = 0$ or $\lim_{t\to\infty}[z_v(t) \times \dot{\boldsymbol{x}}(t)] = 0$. Thus, it is shown that $z_v(t) \times \dot{\boldsymbol{x}}(t) \to 0$ and $z_v(t) \times \boldsymbol{x}(t) \to 0$ as $t \to \infty$ simultaneously. This can be possible only if $z_v(t) = 0$ because every unit-vector $\boldsymbol{x}(t)$ satisfies $\boldsymbol{x}^T(t)\dot{\boldsymbol{x}}(t) = 0$ (i.e., vectors $\boldsymbol{x}(t)$ and $\dot{\boldsymbol{x}}(t)$ remain normal to one another for all t). This completes the proof with the fact that $\lim_{t\to\infty} z_v(t) = 0$ is equivalent to $\lim_{t\to\infty} C^T[\boldsymbol{q}(t)]C[\hat{\boldsymbol{q}}(t)] = I_{3\times 3}$ from Eq. (26). □

In addition, the following important observations are in order.

1) The equilibrium manifold $z_o = 0$ that results whenever $\hat{\boldsymbol{q}}^T(0)\boldsymbol{q}(0) = 0$ happens can actually be shown to be unstable. From Eq. (31), V_e belongs to the closed interval $[0, 1/2]$ with $V_e = 0$ whenever $z_o = \pm 1$ and $V_e = 1/2$ whenever $z_o = 0$. If $z_o(0) \doteq \xi \neq 0$ (i.e., $V_e(0) < 1/2$) with arbitrary small ξ, then $|z_o(t)| \geq |\xi|$ for all $t > 0$ from Eq. (30) and thereby, it is impossible that $z_o(t) \to 0$ as $t \to \infty$. Consequently, equilibrium $z_o = 0$ defines an unstable manifold.
2) The convergence result of Proposition 2 can also be established by the following new Lyapunov-like candidate function V_o defined as follows

$$V_o = \frac{z_v^T z_v}{2\gamma z_o^2}; \qquad \gamma > 0 \tag{34}$$

which is well defined and a bounded function whenever $\hat{q}(0) \in \Psi_s$ or, in other words, $z_o(t) \neq 0$ for all $t \geq 0$. By differentiating V_o with respect to time along Eq. (30), we can show that

$$\begin{aligned}\dot{V}_o &= \frac{1}{\gamma z_o^4}(z_o^2 z_v^T \dot{z}_v - z_o \dot{z}_o z_v^T z_v) \\ &= -|z_v \times x\|^2 - \frac{\|z_v \times x\|^2}{z_o^2} z_v^T z_v \leq -\|z_v \times x\|^2 \end{aligned}\tag{35}$$

We can follow the same steps described in the proof of Proposition 2 and thus, the asymptotic convergence $C(\hat{q}) \to C(q)$ is again guaranteed.

3) The modification from V_e to V_o is a crucial development in the context of the results being sought in this paper because this new construction for candidate function V_o helps in the production of the term $\|z_v \times x\|^2$ in $\dot{V}_o(t)$ as seen from Eq. (35) which is independent of z_o. Detailed derivations to be presented in the sequel will further illustrate how this modification can help compensate for the closed-loop attitude tracking and estimation errors when the attitude estimator of Eq. (23) is adopted as part of a feedback control algorithm.
4) From Proposition 2, the group of initial conditions $\hat{q}(0)$ corresponding to the condition $\hat{q}(0)^T q(0) = 0$ are categorized as an unstable initial condition [belonging to unstable manifold $z_o(t) = 0$ for all $t \geq 0$] for the attitude estimator. The effects of initializing the attitude estimator while satisfying the condition $\hat{q}(0)^T q(0) = 0$ will be demonstrated in the numerical simulations section.
5) Having the unit-vector reference signal $x(t)$ that drives the observer to be not constant with time is shown to be sufficient for ensuring that the attitude estimate matrix $C(\hat{q})$ converges to the actual attitude matrix $C(q)$, of course whenever $\hat{q}(0) \in \Psi_s$. In effect, this condition ensures satisfaction of "persistence of excitation" so far as the attitude observer design is concerned. More important, verifying whether this condition holds or not is as simple a matter as determining if the reference signal x is constant with time or not.

IV. Separation Property of Observer-Based Attitude Control

Analogous to the separation principle of linear control theory, we develop here a closed-loop stability result for the nonlinear rigid-body attitude tracking control system by carefully combining two separately designed control components: 1) the full-state feedback part (as if the true quaternion is exactly determined), and 2) the attitude observer that estimates the quaternion using measured signals. In the following, we first derive the attitude tracking error dynamics based on the estimated attitude information $\hat{q}$ from the attitude observer Eq. (23).

We start by defining the observer-based attitude tracking error $\delta\hat{q}$ through the following implicit relationship

$$C(\delta\hat{q}) = C(\hat{q})C^T(q_r) \tag{36}$$

Next, we introduce the estimated body frame E in association with the estimated quaternion $\hat{q}$ which satisfies the following relationships:

$$N \overset{\hat{q}}{\to} E \Rightarrow \hat{e} = C(\hat{q})\hat{n} \qquad R \overset{\delta\hat{q}}{\to} E \Rightarrow \hat{e} = C(\delta\hat{q})\hat{r}$$

where $\hat{e}$ represents the unit-vector triad in the estimated body frame E. By following the same steps taken in deriving Eq. (8), the time derivative of Eq. (36) is obtained by

$$\frac{d}{dt}C(\delta\hat{q}) = -S(\delta\hat{\omega} + \gamma y \times \hat{y})C(\delta\hat{q}) \tag{37}$$

where $\delta\hat{\omega}$ is defined by

$$\delta\hat{\omega} = \omega - C(\delta\hat{q})\omega_r \tag{38}$$

The quaternion realization of Eq. (37) is represented as [20]

$$\delta\dot{\hat{q}} = \frac{1}{2}E(\delta\hat{q})(\delta\hat{\omega} + \gamma y \times \hat{y}) \tag{39}$$

Similarly, by taking the time derivative of Eq. (38), and using the Euler differential equation from Eq. (5), we derive the following observer-based angular velocity error dynamics

$$J\delta\dot{\hat{\omega}} = -S(\omega)J\omega + u + J\eta \tag{40}$$

wherein η is defined by

$$\eta = S(\delta\hat{\omega} + \gamma y \times \hat{y})C(\delta\hat{q})\omega_r - C(\delta\hat{q})\dot{\omega}_r \tag{41}$$

It is possible to interpret the overall tracking error dynamics as given by Eqs. (39) and (40), so that our control objective for the attitude tracking problem is to specify the control signal $u(t)$ in such a way to achieve $\lim_{t\to\infty}[\delta\hat{q}_v(t), \delta\hat{\omega}(t)] = 0$ along with $\lim_{t\to\infty}\{C[\hat{q}(t)] - C[q(t)]\} = 0$.

We now present the main contribution of this paper, a separation property of the rigid-body attitude tracking control problems based on the new observer-based attitude tracking control method.

Theorem 1. For the attitude tracking error dynamics described through Eqs. (39) and (40), suppose the control input $u(t)$ is computed by

$$\begin{aligned} u = &-k_v J\delta\hat{\omega} + S(\omega)J\omega - J\eta \\ &- k_p J\left(\alpha\delta\hat{q}_v + \frac{1}{2}[\delta\hat{q}_o I_{3\times 3} + S(\delta\hat{q}_v)](\delta\hat{\omega} + \gamma y \times \hat{y})\right) \end{aligned}\tag{42}$$

where $k_p > 0$, $k_v > 0$, $\alpha = k_p + k_v$. Further, suppose the attitude estimate $\hat{q}(t)$ is computed from Eq. (23) under the assumptions of Proposition 2 governing the reference signal x (i.e., unit-vector $x(t)$ is not a constant with respect and has bounded time derivatives). If $\hat{q}(0)$ satisfies $\hat{q}(0)^T q(0) \neq 0$, then the attitude tracking control objective $\lim_{t\to\infty}[\delta q_v(t), \delta\omega(t)] = 0$ is guaranteed (almost) globally and asymptotically. On the other hand, if the attitude estimator is initiated such that the initial value of the estimated quaternion $\hat{q}(0)$ satisfies $\hat{q}(0)^T q(0) = 0$, then all closed-loop signals remain bounded with no further assurance of asymptotic convergence of the tracking errors.

Proof. Suppose $\hat{q}^T(0)q(0) \neq 0$ and accordingly, $z_o(t) \neq 0$ for all $t \geq 0$. We consider a composite Lyapunov-like candidate function that is motivated from the results and discussion presented under Propositions 1 and 2 and is defined as follows

$$V = V_c + V_o = \underbrace{\frac{1}{2}\hat{\omega}_f^T\hat{\omega}_f + \left[(\delta\hat{q}_o - 1)^2 + \delta\hat{q}_v^T\delta\hat{q}_v\right]}_{V_c\text{(controller part)}} + \underbrace{\frac{\lambda z_v^T z_v}{2z_o^2}}_{V_o\text{(observer part)}} \tag{43}$$

where $\lambda = 16\gamma^2/k_p$. The construction V_c is motivated by Eq. (18) and involves a new filter state $\hat{\omega}_f$ driven by the angular velocity error $\delta\hat{\omega}$ in Eq. (38) as follows

$$\dot{\hat{\omega}}_f = -\alpha\hat{\omega}_f + \delta\hat{\omega} \tag{44}$$

Differentiating both sides of Eq. (44) with respect to time and making a substitution from Eqs. (40) and (42), filtered error dynamics can be seen to satisfy

$$\dot{\hat{\boldsymbol{\omega}}}_f = -k_v\hat{\boldsymbol{\omega}}_f - k_p\delta\hat{\boldsymbol{\omega}} \tag{45}$$

wherein we have already neglected an exponentially decaying term which can be justified though arguments identical to those presented earlier that led to establishment of Eq. (20). The time derivative of the Lyapunov-like candidate function V taken along with Eqs. (28), (39), and (45) yields the following

$$\dot{V} = \dot{V}_c + \dot{V}_o = -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + \gamma\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T(\boldsymbol{y}\times\hat{\boldsymbol{y}}) - \lambda\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2 - \lambda\frac{\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2}{z_o^2} \tag{46}$$

We can further simplify $\dot{V}$ from Eq. (46) to obtain

$$\begin{aligned}\dot{V} &= -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + \gamma\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T C(\boldsymbol{q})[2z_o(\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x})\times\boldsymbol{x} \\ &\quad - 2(\boldsymbol{x}^T\boldsymbol{z}_{\boldsymbol{v}})(\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x})] - \lambda\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2 - \lambda\frac{\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2}{z_o^2} \\ &\leqslant -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + 4\gamma\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\| - \lambda\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2 \\ &\leqslant -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - \frac{k_p}{2}\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 - \frac{\lambda}{2}\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2 \\ &\quad - \frac{k_p}{2}\left[\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 - \frac{8\gamma}{k_p}\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\| + \frac{\lambda}{k_p}\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2\right] \\ &\leqslant -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - \frac{k_p}{2}\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 - \frac{\lambda}{2}\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2\end{aligned} \tag{47}$$

wherein the last inequality follows from the choice of $\lambda = 16\gamma^2/k_p$. Following essentially same steps as in the proof of Proposition 2, we now can conclude boundedness of all the closed-loop signals. Further, it also follows that $\dot{\hat{\boldsymbol{\omega}}}_f$, $\delta\dot{\hat{\boldsymbol{q}}}_{\boldsymbol{v}}$, and $\dot{\boldsymbol{z}}_{\boldsymbol{v}}$ are $\mathcal{L}_\infty$. In addition, integrating both sides of Eq. (47), we readily establish $\|\hat{\boldsymbol{\omega}}_f\|$, $\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|$, and $\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|$ are $\mathcal{L}_2\cap\mathcal{L}_\infty$. Thus, by Barbalat's lemma, $[\|\hat{\boldsymbol{\omega}}_f(t)\|, \|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}(t)\|, \|\boldsymbol{z}_{\boldsymbol{v}}(t)\times\boldsymbol{x}(t)\|]\to 0$ as $t\to\infty$ which is equivalent to $[\delta\hat{\boldsymbol{\omega}}(t), \delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}(t), \boldsymbol{z}_{\boldsymbol{v}}(t)]\to 0$ as $t\to\infty$ given the filter definition Eq. (44) and the fact that the unit-vector reference signal $\boldsymbol{x}$ is not constant with time. Consequently, we guarantee that $\lim_{t\to\infty}[\delta\boldsymbol{\omega}(t), \delta\boldsymbol{q}_{\boldsymbol{v}}(t)] = 0$ along with $\lim_{t\to\infty}\{C[\hat{\boldsymbol{q}}(t)] - C[\boldsymbol{q}(t)]\} = 0$.

Next, for the case $\hat{\boldsymbol{q}}^T(0)\boldsymbol{q}(0) = 0$, [i.e., $z_o(0) = 0$], we consider the following lower-bounded Lyapunov-like candidate function

$$V_u = V_c + V_e = \underbrace{\frac{1}{2}\hat{\boldsymbol{\omega}}_f^T\hat{\boldsymbol{\omega}}_f + \left[(\delta\hat{q}_o - 1)^2 + \delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\right]}_{V_c} + \underbrace{\frac{\boldsymbol{z}_{\boldsymbol{v}}^T\boldsymbol{z}_{\boldsymbol{v}}}{2}}_{V_e} \tag{48}$$

Then the time derivative of V_u is given as follows

$$\dot{V}_u(t) = -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + \gamma\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T(\boldsymbol{y}\times\hat{\boldsymbol{y}}) - \gamma z_o^2\|\boldsymbol{z}_{\boldsymbol{v}}\times\boldsymbol{x}\|^2 \tag{49}$$

Because $z_o(t) = 0$ for all $t \geq 0$ as long as $z_o(0) = 0$ from Eq. (30), we can simplify Eq. (49) further as follows

$$\begin{aligned}\dot{V}_u(t) &= -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 - k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + \gamma\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T(\boldsymbol{y}\times\hat{\boldsymbol{y}}) \\ &\leq -k_v\|\hat{\boldsymbol{\omega}}_f\|^2 + c_1; \quad c_1 \doteq \sup_{t\geq 0}\{k_p\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\|^2 + \gamma\|\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T(\boldsymbol{y}\times\hat{\boldsymbol{y}})\|\}\end{aligned} \tag{50}$$

where c_1 is the well-defined finite positive constant consisting of norms of the bounded quaternion element $\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}$ and the quantity $\boldsymbol{y}\times\hat{\boldsymbol{y}}$ (vector cross product between unit vectors). Similarly, from Eq. (48), we can represent the upper bounding function for V_u as follows

$$\begin{aligned}V_u &\leq \frac{1}{2}\|\hat{\boldsymbol{\omega}}_f\|^2 + c_2 \\ c_2 &\doteq \sup_{t\geq 0}\left\{\left[(\delta\hat{q}_o - 1)^2 + \delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}^T\delta\hat{\boldsymbol{q}}_{\boldsymbol{v}}\right] + \frac{\boldsymbol{z}_{\boldsymbol{v}}^T\boldsymbol{z}_{\boldsymbol{v}}}{2}\right\} \leq \frac{5}{2}\end{aligned} \tag{51}$$

Based on Eqs. (50) and (51), $\dot{V}_u$ is bounded from above by

$$\dot{V}_u \leq -2k_vV_u + c_3; \qquad c_3 = 2k_vc_2 + c_1 \tag{52}$$

which shows that V_u is bounded and in turn, $\hat{\boldsymbol{\omega}}_f$ is bounded from Eq. (48). Because $\hat{\boldsymbol{\omega}}_f$ is bounded, $\delta\hat{\boldsymbol{\omega}}$ is also bounded from the definition of filter dynamics in Eq. (44) because the rest of the closed-loop signals are already bounded quaternion elements ($\delta\hat{\boldsymbol{q}}$ and $\boldsymbol{z}$). Further, $\boldsymbol{\omega}$ is bounded from Eq. (38). Therefore, every closed-loop signal is bounded and likewise, the control torque is bounded from Eq. (42). This completes the proof for all stated assertions. □

V. Numerical Simulations

To demonstrate the performance of the attitude estimator (observer) combined with the proposed controller, we perform simulations. The reference signal $\boldsymbol{x}(t)$ for the attitude observer is taken to be $\boldsymbol{x}(t) = [\cos t, \sin t, 0]^T$. The commanded attitude reference trajectory taken from the literature [9] is given as $\boldsymbol{\omega}_r = \omega_r(t)\times[1,1,1]^T$ where $\omega_r(t) = 0.3\cos t(1 - e^{-0.01t^2}) + (0.08\pi + 0.006\sin t)te^{-0.01t^2}$. We assume initial conditions $\boldsymbol{q}(0) = [(2\sqrt{2})/3, 1/(3\sqrt{3}), 1/(3\sqrt{3}), 1/(3\sqrt{3})]^T$, $\hat{\boldsymbol{q}}(0) = [\sqrt{3}/2, 1/(2\sqrt{3}), 1/(2\sqrt{3}), 1/(2\sqrt{3})]^T$, $\boldsymbol{q}_r(0) = [1,0,0,0]^T$, and $\boldsymbol{\omega}(0) = [0,0,0]$ (i.e., the body is initially at rest). The inertia matrix J is chosen as

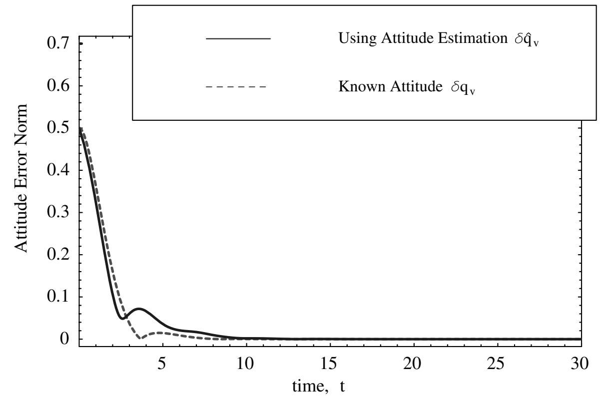
$$J = \begin{bmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 17 & 1.4 \\ 0.9 & 1.4 & 15 \end{bmatrix} \tag{53}$$

and control gains γ, k_p, and k_v are all set to unity.

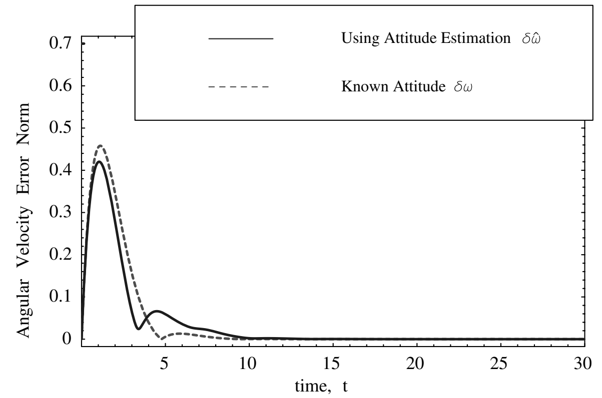
In Fig. 1, we show results from two sets of simulations wherein, in one case, the controller is given by Eq. (42) by using the attitude estimator, and the second case corresponds to the controller from Eq. (13) that assumes instantaneous availability of the actual true quaternion $\boldsymbol{q}(t)$. It is clear from Figs. 1a and 1b that the overall performance (speed of tracking error convergence) for the controller that assumes availability of the true attitude quaternion is slightly faster compared with the controller that needs to perform the additional task of attitude estimation while attending to the underlying tracking objective. Of course, this is quite reasonable and bound to happen because the controller based on the attitude estimator needs to apply additional effort toward compensating the difference between the estimated attitude value and the true attitude value as shown in Fig. 1c (note that origin in this plot is slightly offset to the "first quadrant" to help better visualization) and Fig. 1d. Once the attitude estimates converge to the true attitude values, the control norms of either controller become virtually identical and they remain essentially nonzero to ensure the body maintains the prescribed attitude tracking motion. This happens after about 10 s into the simulation and is shown in Fig. 1d.

Next, we simulate the case in which the initial condition for the attitude estimator $\hat{\boldsymbol{q}}(0)$ is deliberately chosen such that the condition $z_o(0) = 0$ is identically satisfied. In Fig. 2, we show the simulation results that compare the closed-loop performance of the controllers from Eqs. (13) and (42). It is evident from these results that the attitude estimation error converges to zero after about 60 s, which is much slower compared with the convergence rates documented in Fig. 1. Correspondingly, control efforts commanded by the controller to enable attitude tracking are also greater, as seen in Fig. 2c. Because the initial condition for the estimator in these simulations corresponds to an unstable manifold [17], the attitude estimator expectedly converges to the actual attitude state within our simulations after going through an initial transient.

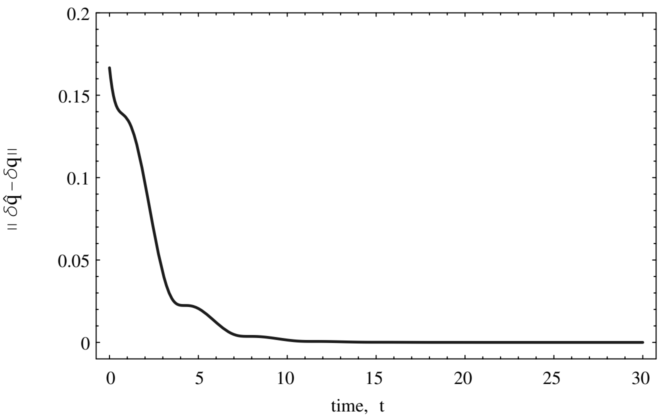
It needs to be emphasized that there is no special significance to the time instant $t\sim 50$ s except for the fact that in the case of this particular simulation, it took about 50 s for the numerical integration errors to sufficiently build up and drive the attitude estimator away from the unstable equilibrium manifold corresponding to $z_o = 0$. It is perhaps more important to note that during the time the attitude estimator is stuck within the unstable equilibrium manifold (for



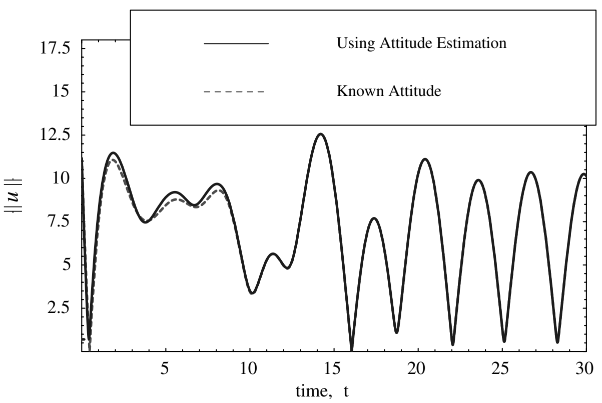
a) **Quaternion attitude error**



b) **Angular velocity error**

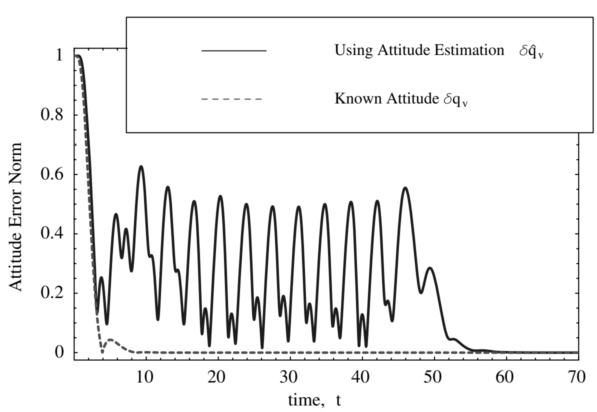


c) **Attitude estimator performance**

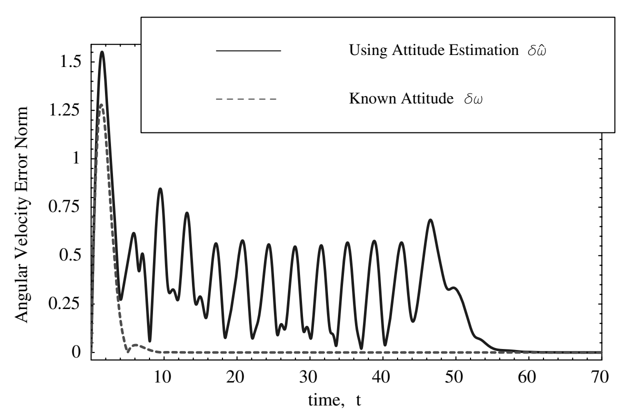


d) **Control torque**

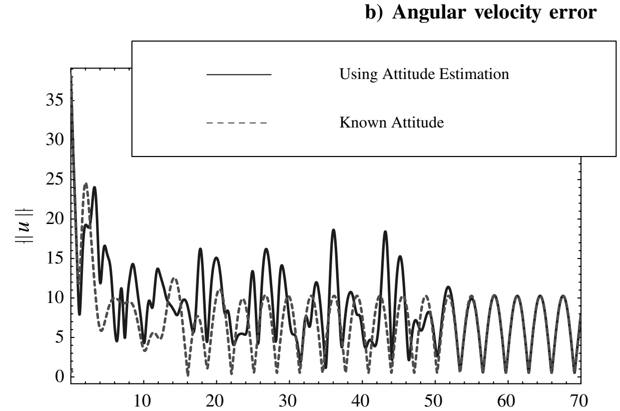
Fig. 1 Closed-loop simulation results comparing performance of the controller from Eq. (42) with controller from Eq. (13).



a) **Quaternion attitude error**



b) **Angular velocity error**



c) **Control torque**

Fig. 2 Closed-loop simulation results comparing performance of the controller from Eq. (42) with controller from Eq. (13). The attitude estimator is initialized in such a way that $q(0)$ exactly satisfies condition $\hat{q}(0)^T q(0) = 0$.

$t \leq 50$ s), the control torque and all other closed-loop trajectories remain bounded just as it is assured by our theoretical results.

VI. Conclusions

A vast majority of closed-loop stability results governing attitude control designs assume direct availability of the attitude (quaternion) variables for feedback implementations. In practice, however, there exists no physical sensor that permits direct and exact measurement of the quaternion vector. In effect, the attitude quaternion is always extracted from accompanying filters/observers that are driven by measurements from rate-gyros, sun sensors, star sensors, earth sensors, magnetometers, and a host of other sensor candidates, essentially depending upon the particular application. In terms of theory, given that the governing dynamics are nonlinear and time-varying, any stability result obtained assumed exact measurement of the quaternion vector needs to be reestablished all over again once the estimated quaternion is engaged within the feedback control law.

In this paper, we present a new convergence analysis process for the quaternion estimator that ensures asymptotic convergence to the true attitude quaternion. Further, we are able to successfully couple the attitude observer with a new attitude stabilization controller to ultimately establish a separation property delivering global stability for the composite closed-loop system. In this process, we also show a novel approach for surmounting the technical difficulties due to nonstrict Lyapunov functions within existing formulations. In essence, our approach involves synthesis of a novel Lyapunov function construction that is strict and derived in terms of precisely characterized linear dynamic filters. More specifically, our stability analysis is accomplished without relying upon cancellation of terms in the Lyapunov function derivative. Instead, we achieve the necessary closed-loop stability assurances through domination of indefinite terms which result from the attitude estimator implementation by making use of negative definite terms made possible by the strict Lyapunov constructions.

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